BOUNDS ON THE PRINCIPAL FREQUENCY OF THE p-LAPLACIAN

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ABSTRACT. In this paper, we extend some known bounds for the principal frequency of the Laplace operator to the p-Laplacian. In particular, we present some results involving the inner radius of a planar domain. As a by-product, we generalize to the p-Laplace operator a lower bound on the size of a nodal set in the planar case. We also discuss the higher dimensional case and present some extensions for the p-Laplacian of results obtained earlier by Maz'ya and Shubin for the usual Laplace operator.

Keywords. Inradius, p-Laplacian, principal frequency, nodal set, capacity, interior capacity radius.

1. Overview of the p-Laplacian

1.1. **Physical models involving the p-Laplacian.** Let Ω be a bounded open subset of $n \geq 2$-dimensional Euclidean space $\mathbb{R}^n$. For $1 < p < \infty$, the $p-$Laplacian of a function f on Ω is defined by $\Delta_p f = -\text{div}(|\nabla f|^{p-2}\nabla f)$ for suitable f. The p-Laplacian can be used to model the flow of a fluid through porous media in turbulent regime (see for instance [DHT, DTh]), the glacier's ice when treated as a non-Newtonian fluid with a nonlinear relationship between the rate deformation tensor and the deviatoric stress tensor (see [GR]), and in the Hele-Shaw approximation (a moving boundary problem involving the p-Laplacian; see [KMC]).

Let us present a model, well known in the case of the Laplace operator, which remains very useful to understand the physical meaning behind some inequalities that we shall prove, in particular those involving the inner radius of Ω. The nonlinearity of the p-Laplacian is often used to reflect the impact of non ideal material to the usual vibrating homogeneous elastic membrane, modeled by the Laplace operator. Thus, the following is used to describe, for instance, a nonlinear elastic membrane under the load f,

$$\begin{aligned} -\Delta_p(u) &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{1.1.1}$$

The solution u_f stands for the deformation of the membrane from the rest position (see [CEP, Si]). In particular, if its deformation energy is given by $\int_\Omega |\nabla u|^p dx$, then a minimizer of the Rayleigh quotient,

$$\frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega |u|^p dx},$$

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on $W_0^{1,p}(\Omega)$ satisfies $-\Delta_p(u) = \lambda_{1,p}|u|^{p-2}u$ in Ω, where $\lambda_{1,p}$ is usually referred as the principal frequency of the vibrating non elastic membrane.

1.2. **The eigenvalue problem for the p-Laplacian.** For $1 < p < \infty$, we study the following eigenvalue problem:

$$\Delta_p u + \lambda|u|^{p-2}u = 0 \text{ in } \Omega, \tag{1.2.1}$$

where we impose the Dirichlet boundary condition and consider λ to be the real spectral parameter. We say that λ is an eigenvalue of $-\Delta_p$ if (1.2.1) has a nontrivial weak solution $u_\lambda \in W_0^{1,p}(\Omega)$, namely if

$$\int_\Omega |\nabla u_\lambda|^{p-2}\nabla u_\lambda \cdot \nabla v - \lambda \int_\Omega |u_\lambda|^{p-2}u_\lambda v = 0, \tag{1.2.2}$$

holds for any $v \in W_0^{1,p}(\Omega)$. The function u_λ is then called an eigenfunction of $-\Delta_p$ associated to λ.

If $n \geq 2$ and $p = 2$, it is well known that one can obtain an increasing sequence of eigenvalues tending to $+\infty$ via the Rayleigh-Ritz method. Moreover, those are all the eigenvalues of Δ_2 and linearity is crucial to show that.

For the general case, it is known that the first eigenvalue of the Dirichlet eigenvalue problem of the p-Laplace operator, denoted by $\lambda_{1,p}$, admits the following variational characterization,

$$\lambda_{1,p} = \min_{0 \neq u \in W_0^{1,p}(\Omega)} \left\{ \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega |u|^p dx} \right\}, \tag{1.2.3}$$

and is simple and isolated (there is no sequence of eigenvalues such that $\lambda_{k,p}$ tends to $\lambda_{1,p}$; see [Lind2]). Moreover, the eigenfunction u_1 associated to $\lambda_{1,p}$ does not change sign, and it is the only such eigenfunction (a proof can be found in [Lind2]). As for $\lambda_{2,p} > \lambda_{1,p}$, it allows a min-max characterization and every eigenfunction associated to $\lambda_{2,p}$ changes sign only once in Ω (it was first shown in [ACFK], but one can also check [D1]). It is not known if $\lambda_{2,p}$ is isolated. Via for instance Lyusternick-Schnirelmann maximum principle, it is possible to construct $\lambda_{k,p}$ for $k \geq 3$ and hence obtain an increasing sequence of eigenvalues of (1.2.1). There exist other variational characterizations of the Δ_p eigenvalues, but it always remains to show that these exhaust the spectrum of Δ_p.

2. Lower bounds involving the inradius of a planar domain

2.1. **Known bounds on the principal frequency.** Using the domain monotonicity property, it is an easy matter to obtain an upper bound for the principal frequency of the p-Laplacian. Indeed, for $B_r \subset \Omega$, we have that $\lambda_{1,p}(\Omega) \leq \lambda_{1,p}(B_r) = \frac{\lambda_{1,p}(B_1)}{r^p}$. Therefore, if we consider the largest ball that can be included in Ω, we get

$$\rho_\Omega \leq \left(\frac{\lambda_{1,p}(B_1)}{\lambda_{1,p}(\Omega)} \right)^{\frac{1}{p}}, \tag{2.1.1}$$

where $\rho_\Omega := \sup\{r : \exists B_r \subset \Omega\}$. It is important to note that unlike the case $p = 2$ corresponding to the Laplace operator, there are no explicit formulas for $\lambda_{1,p}$ of a ball.

Lower bounds involving the principal frequency, however, have always been a greater challenge. Nevertheless, some results are known. Let us start by recalling that the classical Faber-Krahn inequality can be adapted to the p-Laplacian (as noted in [Lind1, p. 224] and [Hua, p. 3353], it is a consequence of results in [Ka]) : among all domains of given n-dimensional volume, the ball minimizes every $\lambda_{1,p}$. In other words, we have that $\lambda_{1,p}(\Omega) \geq \lambda_{1,p}(\Omega^*)$, where Ω^* stands for the n-dimensional ball of same volume as Ω.

For the Laplacian, lower bounds of the type

$$\lambda_{1,2}(\Omega) \geq \alpha_{n,2}\ \rho_\Omega^{-2},$$

where $\alpha_{n,2} > 0$ is a positive constant, have been studied extensively.

If $n = 2$, the first result proved in that direction is due to J. Hersch in [He], and for convex simply connected planar domains, gives the constant $\alpha_{2,2} = \frac{\pi^2}{4}$. This result was extended by E. Makai in [Mak]. For all simply connected domains, he obtained the constant $\alpha_{2,2} = \frac{1}{4}$.

An extension of Makai's method for the pseudo p-Laplacian was studied in [B] and lead to a similar lower bound for simply-connected convex planar domain.

By a different approach, W. K. Hayman also obtained a bound for simply-connected domains with, however, the constant, $\alpha_{2,2} = 1/900$. R. Osserman (see [O1, O2, O3]) later improved that result to $\alpha_{2,2} = 1/4$, the same constant obtained by Makai, and also extended the inradius bound for domains of connectivity $k \geq 2$; a result that was improved by [Cr].

2.2. **Generalized inradius bounds for the p-Laplacian.** We present some lower bounds for the principal frequency involving the inradius of the domain,

$$\lambda_{1,p}(\Omega) \geq \alpha_{n,p}\ \rho_\Omega^{-p},$$

by adapting proofs obtained for the usual Laplacian. To do so, we need two main ingredients: a modified Cheeger-type inequality (see the original result for the Laplacian in [Che] and a generalized version for the p-Laplacian can be found in [KF]), and a geometric inequality relating the ratio of the length of the boundary of a domain and its area with its inradius. Regarding the modified Cheeger inequality, it consists of an adapted version of a result in [O4]:

Lemma 2.2.1. *Let D be a domain homeomorphic to a planar domain of finite connectivity k. Let F_k be the family of relatively compact subdomains of D having smooth boundary and connectivity at most k. Let*

$$h_k(D) = \inf_{D' \in F_k} \frac{L'}{A'}, \tag{2.2.2}$$

where A' is the area of D' and L' is the length of its boundary. Then,

$$\lambda_{1,p}(D) \geq \left(\frac{h_k(D)}{p}\right)^p. \tag{2.2.3}$$

The crucial point of Lemma 2.2.1 resides in the fact that the Cheeger constant is computed among subdomains that have a connectivity of at most the connectivity of the domain, which allows us to use geometric inequalities accordingly.

We start by proving the following an extension of Osserman-Croke's result for the p-Laplacian. We actually generalize a stronger result that was implicit in [Mak] and [O1], but made explicit in [G]. Instead of considering the inradius, we use the reduced inner radius, which is defined as follow, $\tilde{\rho}_\Omega := \frac{\rho_\Omega}{1+\frac{\pi\rho_\Omega^2}{|\Omega|}}$. Notice that $\frac{\rho_\Omega}{2} < \tilde{\rho}_\Omega < \rho_\Omega$.

The first main result consists of extending classical planar inradius bounds of the Laplace operator to the case of the p-Laplacian:

Theorem 2.2.4. *Let Ω be a domain in $\mathbb{R}^2$. If Ω is simply connected, then*

$$\lambda_{1,p}(\Omega) \geq \left(\frac{1}{p\ \tilde{\rho}_\Omega}\right)^p. \tag{2.2.5}$$

If Ω is of connectivity $k \geq 2$, then

$$\lambda_{1,p}(\Omega) \geq \frac{2^{p/2}}{k^{p/2}p^p\ \rho_\Omega^p}. \tag{2.2.6}$$

It is hard to say whether these bounds are optimal, since we can not compute eigenfunctions and eigenvalues explicitly on simple domains, unlike in the case of the usual Laplacian. In that situation, instead of the constant $\frac{1}{4}$ given by (2.2.4), the better constant ≈ 0.6197 was found using probabilistic methods in [BaC].

Remark 2.2.7. For a bounded domain Ω in $\mathbb{R}^n$, the ground state problem associated to the ∞-Laplacian is the following

$$\max\left\{\lambda_{1,\infty} - |\nabla \log u|, \Delta_\infty u\right\} = 0, \tag{2.2.8}$$

where $\Delta_\infty u := \sum_{i,j=1}^n \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j}$. It is a notable feature that $\lambda_{1,\infty} = \frac{1}{\rho_\Omega}$, i.e. the value of $\lambda_{1,\infty}$ can immediately be read off the geometry of Ω, without any topological assumptions on Ω (see [JL] and reference therein for additional details).

2.3. **The limiting case p=1.** As $p \to 1$, the limit equation formally reads

$$\begin{aligned} -\,div\left(\frac{\nabla u}{|\nabla u|}\right) &= \lambda_{1,1}(\Omega) &\quad \text{in } \Omega,\\ u &= 0 &\quad \text{on } \partial\Omega, \end{aligned} \tag{2.3.1}$$

where $\lambda_{1,1}(\Omega) := \lim_{p\to 1^+} \lambda_{1,p}(\Omega) = h(\Omega)$ (see [KF, KS] for instance). Here, Ω is assumed to be smooth enough. If we restrict the subdomains considered in the computation of $h(\Omega)$ to smooth simply connected ones, we thus get

Proposition 2.3.2. *If Ω is a simply connected planar domain, then*

$$\lambda_{1,1}(\Omega) \geq \frac{1}{\tilde{\rho}_\Omega}. \tag{2.3.3}$$

2.4. **A bound on the size of the nodal set in the planar case.** This subsection is devoted to the study of the size of the nodal set $\mathcal{Z}_\lambda = \{x \in \Omega : u_\lambda(x) = 0\}$ of an eigenfunction u_λ of (1.2.1). Yau's Conjecture (see [Y]) asserts that the size of nodal sets of an eigenfunction u_λ is comparable to λ^2.

Donnelly and Fefferman (see [DF]) proved Yau's Conjecture for real analytic manifolds. However, if one assumes only that (M, g) is smooth, Yau's

Conjecture remains partially open. In the planar case, it is known that $\mathcal{H}^1(\mathcal{Z}_\lambda) \geq C_1\lambda^{1/2}$, where $\mathcal{H}^1$ stands for the 1-dimensional Hausdorff measure (see (see [Bru]). If $n \geq 3$, lower bounds were obtained, see recent works of Sogge-Zelditch in [SZ], Colding-Minicozzi in [CM], and Mangoubi in [M].

We generalize the lower bound in the planar case for the p-laplacian on a planar bounded domain. However, the situation is slightly different from the one for the Laplace operator mainly since it is still not known whether the interior of $\mathcal{Z}_\lambda$ is empty or not (see for instance [D1, GM]). The result is the following:

Theorem 2.4.1. *Let Ω be a planar bounded domain. If λ is large enough, there exists a positive constant C such that*

$$\mathcal{H}^1(\mathcal{Z}_\lambda) \geq C\lambda^{1/p}. \tag{2.4.2}$$

In order to prove the previous theorem, we need to start by proving the analog of a classical result of the Dirichlet Laplacian stating that every eigenfunction must vanish in a ball of radius comparable to the wavelength:

Proposition 2.4.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Consider any eigenfunction of* (1.2.1)*. Let $R = C\lambda^{\frac{1}{p}}$, where $C > \lambda_{1,p}(B_1)$, the first eigenvalue of a ball of radius* 1*. Then, u_λ vanishes in any ball of radius R.*

3. Lower bounds involving the inradius in higher dimensions

3.1. **Convex domains in $\mathbb{R}^n$.** For the Laplace operator, if $n \geq 3$, bounds of the type,

$$\lambda_{1,2}(\Omega) \geq \alpha_n\ \rho_\Omega^2,$$

are generally not possible to obtain, even if Ω is assumed to be simply connected. In [Hay], Hayman points out that if A is a ball with many narrow, inward pointing spikes removed from it, then $\lambda_{1,2}(A) = \lambda_{1,2}(\text{Ball})$, but the inradius of A will tend to 0. This observation remains valid in the case $1 < p < \infty$.

One way to avoid such difficulty is to consider convex domains in $\mathbb{R}^n$. Indeed, since punctured domains are not convex, the issue described above is avoided, and we can prove the following:

Proposition 3.1.1. *If Ω be a convex body in $\mathbb{R}^n$, then the following inequality,*

$$\lambda_{1,p}(\Omega) \geq \left(\frac{1}{p\rho_\Omega}\right)^p, \tag{3.1.2}$$

holds.

The proof is based on two key facts (see [O3, p. 26]), namely that the inequality,

$$|\partial\Omega| \geq h(\Omega)|\Omega|,$$

is required to be true only for subdomains bounded by level surfaces of the first eigenfunction of Ω (see the proof of Lemma 2.2.1), and that if Ω is convex, then those subdomains Ω' are also convex (see [BL, Theorem 1.13]). Recalling that $\frac{1}{\rho_{\Omega'}} \geq \frac{1}{\rho_\Omega}$, the proposition then follows easily.

3.2. **Lieb's and Maz'ya-Shubin's approaches to the inradius problem.** E. Lieb obtained a similar lower bound by relaxing the condition that the ball has to be completely included in Ω. By doing so, one can also relax some hypotheses on Ω. Instead, throughout this section, we shall only assume that Ω is an open subset of $\mathbb{R}^n$. No assumptions on the boundedness or on the smoothness of the boundary of Ω are required. We denote the bottom of the spectrum of $-\Delta_p$ by $\lambda_{1,p}(\Omega)$. In the case when Ω is a bounded domain, $\lambda_{1,p}(\Omega)$ corresponds to the lowest eigenvalue of $-\Delta_p$ with Dirichlet boundary condition as defined in (1.2.3). In the general case, we write

$$\lambda_{1,p}(\Omega) = \inf_{u \in C_0^\infty(\Omega)} \left\{ \frac{\int_\Omega |\nabla|^p dx}{\int_\Omega |u|^p dx} \right\}. \tag{3.2.1}$$

Lieb's method to get such a bound is to allow a fixed fraction $\alpha \in (0,1)$ of the Lebesgue measure of the ball to stick out of Ω. The result can be found in [L, p. 446] and states that for a fixed $\alpha \in (0,1)$, if

$$\sigma_{n,p}(\alpha) = \lambda_{1,p}(B_r)\delta_n^{p/n}(\alpha^{-1/n} - 1)^p,$$

where $\delta_n = \frac{r^n}{|B_r|}$, then

$$\lambda_{1,p}(\Omega) \geq \frac{\sigma_{n,p}(\alpha)}{(r^L_{\Omega,\alpha})^p}, \tag{3.2.2}$$

where $r^L_{\Omega,\alpha} = \sup\{r : \exists B_r \text{ such that } |B_r \setminus \Omega| \leq \alpha |B_r|\}$. Notice that the constant in the lower bound is not totally explicit since it depends on $\lambda_{1,p}(B_1)$, which is not known explicitly for $p \neq 2$.

Maz'ya and Shubin obtained a similar bound, but instead of using Lebesgue measure, they considered the Wiener capacity. The goal of this section is to generalize results of [MS] to the p-Laplacian and to do so, we mainly use results stated in [Maz] and simplify the approach used in [MS], while losing the explicit constants in the bounds. Let us define the Wiener harmonic p-capacity for a compact set $F \subset \mathbb{R}^n$, where $n > 2$,

$$\operatorname{cap}_p(F) = \left\{ \int_{\mathbb{R}^n} |\nabla u|^p, u \in \operatorname{Lip}(\mathbb{R}^n), u|_F \equiv 0, 0 \leq u \leq 1 \right\}. \tag{3.2.3}$$

Fix $\gamma \in (0,1)$. A compact set $F \subset B_r$ is said to be (p,γ)-negligible if

$$\operatorname{cap}_p(F) \leq \gamma \operatorname{cap}_p(\overline{B}_r). \tag{3.2.4}$$

Again, if $p = 2$, our definitions are coherent with the definitions given in [MS]. We are ready to state the main result of this section and its corollaries:

Theorem 3.2.5. *Let $K_1(\gamma,n,p), K_2(\gamma,n,p)$ be positive constants that depend only on γ, n, p. If $1 < p < n$, we have the following inequality,*

$$K_1(\gamma,n,p) r_{\Omega,\gamma}^{-p} \leq \lambda_{1,p}(\Omega) \leq K_2(\gamma,n,p) r_{\Omega,\gamma}^{-p}, \tag{3.2.6}$$

where $r_{\Omega,\gamma} = \sup\left\{r : \exists B_r, \overline{B}_r \setminus \Omega \text{ is } (p,\gamma) - \text{negligible}\right\}$ is the interior p-capacity radius.

A direct application of Theorem 3.2.5 is the following:

Corollary 3.2.7. *If $1 < p < n$, $\lambda_{1,p}(\Omega) > 0 \iff r_{\Omega,\gamma} < +\infty$.*

Corollary 3.2.7 gives a necessary and sufficient condition of strict positivity of the operator $-\Delta_p$ with Dirichlet boundary conditions. For instance, let Ω be the complement of any Cartesian grid in $\mathbb{R}^{n\geq 3}$ (this example is adapted from [Maz, p. 789]. Clearly, for any $\gamma \in (0,1)$, $r_{\Omega,\gamma} = +\infty$; thus, $\lambda_{1,p}(\Omega) = 0$. However, if Ω is a strip, $r_{\Omega,\gamma} < +\infty$, implying that $\lambda_{1,p}(\Omega) > 0$.

Note that since $\lambda_{1,p}(\Omega) > 0$ does not depend on γ, we immediately get the following:

Corollary 3.2.8. *If $1 < p < n$, the conditions $r_{\Omega,\gamma} < +\infty$ for different γ's are all equivalent.*

Also, one can show that for $1 < p < n$, the lower bound given by Theorem 3.2.5 implies the lower bound obtained earlier by Lieb, (3.2.2). To do so, one needs to use an isocapacity inequality that can be found in [Maz, Section 2.2.3], stating that if F be a compact subset of $\mathbb{R}^n$, then

$$\mathrm{cap}_p(F) \geq \omega^{p/n} n^{(n-p)/n} \left(\frac{n-p}{p-1}\right) |F|^{(n-p)/n}, \tag{3.2.9}$$

where equality occurs if and only if F is a ball.

Proposition 3.2.10. *If $\alpha = \gamma^{n/(n-p)}$ and $1 < p < n$, then $r^L_{\Omega,\alpha} \geq r_{\Omega,\gamma}$. In particular, Theorem 3.2.5 implies* (3.2.2).

Proof. Let $C = \omega^{p/n} n^{(n-p)/n} \left(\frac{n-p}{p-1}\right)$ and fix $\gamma \in]0,1[$. Suppose that

$$cap_p(\overline{B}_r \setminus \Omega) \leq \gamma cap_p(\overline{B}_r);$$

therefore, using (3.2.9), we get that

$$\begin{aligned} |\overline{B}_r \setminus \Omega| &\leq C^{-\frac{n}{n-p}} cap_p(\overline{B}_r \setminus \Omega) \\ &\leq C^{-\frac{n}{n-p}} \gamma^{\frac{n}{n-p}} cap_p(\overline{B}_r)^{\frac{n}{n-p}} \\ &= \alpha |\overline{B}_r|, \end{aligned}$$

yielding the desired result. □

4. Lower bounds involving eigenvalues of the p-Laplace operator on manifolds

4.1. **Lower bounds involving the inradius on surfaces.** We can adapt some inradius results of [O1] for surfaces (S, g) with controlled Gaussian curvature to the case of the p-Laplacian:

Theorem 4.1.1. *Let S be a surface. Let $D \subset S$ be a simply connected domain. Denote by K its Gaussian curvature and by β the following quantity $\int_D K^+$, where $K^+ = \max\{K, 0\}$. If $\beta \leq 2\pi$ the following inequality,*

$$\lambda_{1,p}(D) \geq \left(\frac{1}{p\ \tilde{\rho}_{D,\beta}}\right)^p, \tag{4.1.2}$$

holds, where $\tilde{\rho}_{D,\beta} := \dfrac{\rho_D}{1 + \frac{(\pi - \frac{1}{2}\beta)\rho_D^2}{|D|}}$ is the β reduced inner radius.

If the surface has a negative Gaussian curvature, then we have the following:

Theorem 4.1.3. *Let S be a simply connected surface. Let $D \subset S$ be a simply connected domain such that $K \leq -\alpha^2,\ \alpha > 0$, where K stands for its Gaussian curvature. The stronger inequality*

$$\lambda_{1,p}(D) \geq \left(\frac{\alpha \coth(\alpha \rho_D)}{p}\right)^p \tag{4.1.4}$$

holds.
Furthermore, if S is a complete surface, then one has the following,

$$\lambda_{1,p}(D) \geq \left(\frac{\alpha \coth(\alpha R_D)}{p}\right)^p, \tag{4.1.5}$$

where R_D stands for the circumradius of D.

4.2. **Negatively curved manifolds.** The first result, called McKean's theorem (see [McK]) in the case of the Laplace operator, concerns manifolds of negative sectional curvature:

Theorem 4.2.1. *Let (M, g) be a complete and simply connected Riemannian manifold. Let $D \subset M$ be a domain such that its sectional curvature is bounded by $\leq -\alpha^2, \alpha > 0$, then*

$$\lambda_{1,p}(D) \geq \frac{(n-1)^p \alpha^p}{p^p}. \tag{4.2.2}$$

The next result is valid on an arbitrary surface (without additional assumptions, such as being simply connected for instance), but it is only valid for doubly-connected domains.

Proposition 4.2.3. *Let (S, g) be a surface. Let $D \subset S$ be a doubly-connected domain such that $K \leq -1$ where K stands for its Gaussian curvature, then*

$$\lambda_{1,p}(D) \geq \frac{1}{p^p}. \tag{4.2.4}$$

4.3. **Minimal submanifolds in $\mathbb{R}^n$.** It is also possible to prove the following using similar arguments:

Proposition 4.3.1. *Let D be a domain on a m-dimensional minimal submanifold in $\mathbb{R}^n$. If D lies in a ball of radius R, then*

$$\lambda_{1,p}(D) \geq \left(\frac{m}{p}\right)^p. \tag{4.3.2}$$

5. Proofs of main results

5.1. **Proof of Lemma 2.2.1.** By (2.2.2), it follows that if one proves (2.2.3) for all domains in a regular exhaustion of D, then (2.2.3) will also hold for D. Knowing that every finitely-connected domain has a regular exhaustion by domains of the same connectivity, we may assume that D has a smooth boundary. Hence, the p-Laplacian Dirichlet eigenvalue problem admits a solution u_1 corresponding to $\lambda_{1,p}$ and may be chosen without loss of generality such that $u_1 \geq 0$.

Let $g = |u_1|^{p-1}u_1$ and then, Hölder's inequality implies that

$$\begin{aligned}\int_D |\nabla g(x)| dx &= p \int_D |u_1(x)|^{p-1} |\nabla u_1(x)| dx \\ &\leq p ||u_1||_p^{p-1} ||\nabla u_1||_p.\end{aligned}$$

Dividing by $||u_1||_p^p$, one gets

$$\frac{1}{p}\frac{\int_D |\nabla g(x)|dx}{\int_D |g(x)|dx} \leq \frac{||\nabla u_1||_p}{||u_1||_p} = \lambda_{1,p}(D)^{1/p}. \tag{5.1.1}$$

For regular values t of the function g, we define the set

$$D_t = \{y \in D : g(y) > t\}.$$

We want to show that the connectivity of D_t is at most k, since it will imply, by (2.2.2), that

$$L(t) \geq h_k(D)A(t), \tag{5.1.2}$$

for all regular values of t, where $L(t)$ is the length of the boundary of D_t and $A(t)$ the area of D_t.

Since t is regular, the boundary ∂D_t consists of a finite number, say m, of smooth curves $C_1, C_2, ..., C_m$ along which $\nabla g \neq 0$. Let D_t' be any connected component of D_t. If the connectivity of D_t' were greater than k, then the complement of D_t would contain a component lying completely in D of boundary say, C_l. Since $u_1 \in C(D) \cap W^{1,p}(D)$ and

$$\int_D |\nabla u_1|^{p-2}\nabla u_1 \cdot \nabla v \; dx = \lambda_{1,p}\int_D |u_1|^{p-2}u_1 v \; dx \geq 0,$$

for all non negative v in $C_0^\infty(D)$, u_1 is p-superharmonic (see [Lind3, Theorem 5.2]). By definition of p-superharmonicity (again [Lind3, Definition 5.1]), the comparison principle holds. Therefore, since $u_1 = \sqrt[p]{t}$ on C_l, it follows that $u_1^p \geq t$ in the internal region of C_l, which contradicts the fact that the internal region lies in the complement of D_t', so that the function g has to be such that $g < t$.

Now, since the set of singular values of g is a closed set of measure zero by Sard's theorem, its complement is a countable union of open intervals I_n. Thus, we can define the following

$$E_n = \{p \in D : g(p) \in I_n\}.$$

Then, by the coarea formula,

$$\begin{aligned}\int_D |\nabla g(x)| \; dx &\geq \sum_{n=1}^{\infty}\int_{E_n} |\nabla g(x)| \; dx = \sum_{n=1}^{\infty}\int_{I_n} L(t) \; dt \\ &\geq h_k(D)\sum_{n=1}^{\infty}\int_{I_n} A(t) \; dt = h_k(D)\int_D g(x) \; dx. \end{aligned} \tag{5.1.3}$$

Combining (5.1.1) with (5.1.3) completes the proof.

5.2. **Proof of Theorems 2.2.4, 4.1.1, and 4.1.3.** We write down an adaptation of the following lemma which was already proven explicitly in the case $\beta = 0$ in [G]:

Lemma 5.2.1. *If $\alpha \leq \pi$, we have the following*

$$h_k(D) \geq \frac{1}{\tilde{\rho}_{D,\alpha}}, \qquad \textit{where } \tilde{\rho}_{D,\alpha} = \frac{\rho_D}{1 + \frac{\alpha \rho_D^2}{|D|}}. \tag{5.2.2}$$

Proof. The first step is to show the following

$$D' \subset D \implies \tilde{\rho}_{D',\alpha} \leq \tilde{\rho}_{D,\alpha}. \tag{5.2.3}$$

Indeed, first note that for a planar domain A and $\rho_A > 0$, the function $f_A(\rho_A) = \rho_A/(1+\alpha\rho_A^2/|A|)$ is increasing in ρ_A for $\alpha\rho_A^2 \leq |A|$. Notice however that if $\alpha > \pi$, this may no longer be the case. Now, $|D'| \leq |D|$ implies $\tilde{\rho}_{D',\alpha} = f_{D'}(\rho_{D'}) \leq f_D(\rho_{D'})$. Since for $\alpha \leq \pi$, $\alpha\rho_{D'}^2 \leq \alpha\rho_D^2 \leq |D|$, the monotonicity of f_D implies (5.2.3).

The second step is the use of Burago and Zalgaller's inequality that can be found in [BZ] (in Grieser's argument Bonnesen's inequality was used instead, see [Bon], but it is valid only for the specific case $\beta = 0$) and that states that

$$\rho_D|\partial D| \geq |D| + \left(\pi - \frac{1}{2}\beta\right)\rho_D^2 \iff \frac{|\partial D|}{|D|} \geq \frac{1+\frac{\alpha\rho_D^2}{|D|}}{\rho_D} = \frac{1}{\tilde{\rho}_{D,\alpha}}, \tag{5.2.4}$$

where $\alpha = \pi - \frac{1}{2}\beta$, which is indeed always $\leq \pi$. By definition of $h_k(D)$, together with (5.2.3) and (5.2.4), one gets the desired result. □

Proof of Theorems 4.1.1 and 2.2.4. If $\beta \leq 2\pi$, combining Lemma 2.2.1, Lemma 5.2.1, and the fact that $\rho_{\Omega'} \leq \rho_\Omega$ provided that $\Omega' \subset \Omega$, we get the desired result. Note that the specific case $\beta = 0$ implies the first part of Theorem 2.2.4.

In order to adapt the argument to planar domains of connectivity k, one must use a generalized geometric inequality that can be found in [Cr, Theorem 1]. Combining Lemma 2.2.1, [Cr, Theorem 1], and the fact that $\rho_{\Omega'} \leq \rho_\Omega$ provided that $\Omega' \subset \Omega$ yields the desired result. □

Proof of Theorem 4.1.3. In [I, Theorem 1] or in [O3, Theorem 8], we have the following inequality:

$$|\partial D| \geq \frac{\alpha|D|}{\tanh\alpha\rho_D} + \frac{2\pi}{\alpha} tanh\frac{\alpha\rho_D}{2} \geq \alpha|D|\coth(\alpha\rho_D).$$

Combining the previous inequality with Lemma 2.2.1 and the fact that $\rho_{\Omega'} \leq \rho_\Omega$ provided that $\Omega' \subset \Omega$ proves the first part of Theorem 4.1.3. To conclude the proof, one must use [O3, Theorem 8], which states that

$$|\partial D| \geq \alpha|D|\coth(\alpha R_D).$$

Combining the latter inequality with Lemma 2.2.1 and the fact that for any subdomain D' its circumradius satisfies $R' \leq R, \coth(\alpha R') \geq \coth(\alpha R)$ yields the desired result. □

5.3. **Proof of Proposition 2.4.3.** Consider a ball B_R and suppose that $u_\lambda \neq 0$ inside B_R. Since $u_\lambda \neq 0$ in B_R, there exists a nodal domain A of u_λ such that $B_R \subset A$. By (2.1.1), we have that

$$R \leq \rho_A \leq \left(\frac{\lambda_{1,p}(B_1)}{\lambda_{1,p}(A)}\right)^{\frac{1}{p}}.$$

Since the restriction of u_λ corresponds to the first eigenfunction on A, $\lambda_{1,p}(A) = \lambda$; thus, we get that

$$R \leq \left(\frac{\lambda_{1,p}(B_1)}{\lambda}\right)^{\frac{1}{p}},$$

leading to a contradiction.

5.4. **Proof of Theorem 2.4.1.** Since for the p-Laplacian, it is still unclear whether $int(\mathcal{Z}_\lambda)$ is empty or not, we shall have to treat both cases.

Suppose $int(\mathcal{Z}_\lambda) \neq \emptyset$. Recall the following Harnack inequality:

Theorem 5.4.1 (Theorem 1.1 of [Tr]). *Let $K = K(3\rho) \subset \Omega$ be a cube of length 3ρ. Let u_λ be a solution of (1.2.1) associated to the eigenvalue λ such that $0 \leq u_\lambda(x) < M < \infty$ for all $x \in K$, then*

$$\max_{K(\rho)} u_\lambda(x) \leq C \min_{K(\rho)} u_\lambda(x),$$

where C is a positive constant that depends on n, p, ρ and on λ.

Therefore, in order to obtain a case where $int(\mathcal{Z}_\lambda) \neq \emptyset$, u_λ must change sign in every neighborhood of $\partial\{u_\lambda = 0\} \setminus \partial\Omega$ (see Figure 1). Nevertheless, in such case, we have that $\mathcal{H}^1(\mathcal{Z}_\lambda)$.

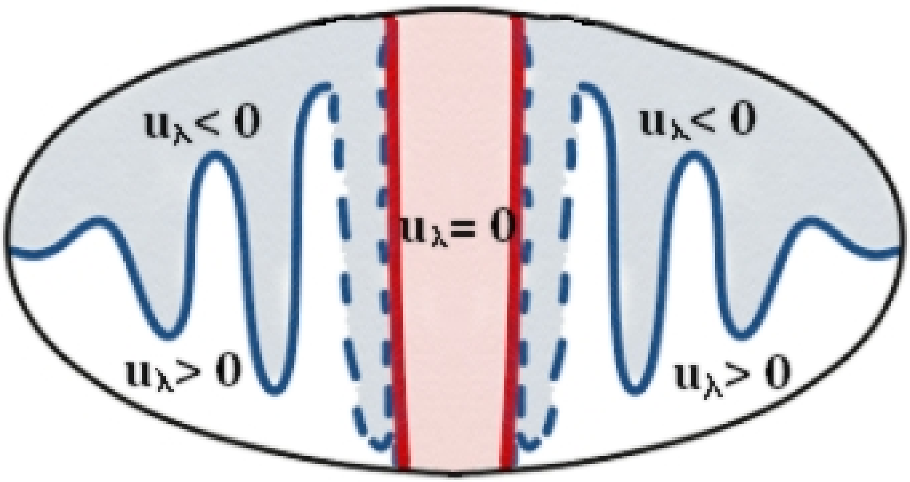
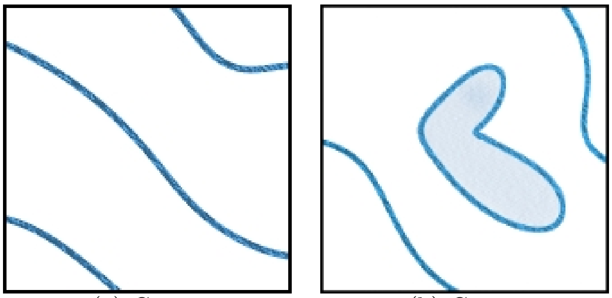


FIGURE 1. The blue nodal line converges to the red line. Notice that the eigenfunction changes sign in every neighborhood of every point on the red line. The example was suggested by P. Drabek (see [D]).

Now, suppose that $int(\mathcal{Z}_\lambda) = \emptyset$. By Proposition 2.4.3, Ω can be split into squares S_c of side $c = Area(\Omega)\lambda^{-1/p}$ such that each square contains a zero of u_λ. Take λ large enough such that the center of each square corresponds to a zero of u_λ. We represent the various cases of nodal lines in a square in Figure 2 and Figure 3.

Notice that the second case of Figure 2 can not occur. Indeed, if it were the case, it would mean that there exists a nodal domain A included the square. Since the eigenfunction restricted to A would correspond to the first one, domain monotonicity would give a contradiction. Indeed, we would have the following

$$\lambda_{1,p}(S_1)\lambda Area(\Omega)^{-p} = \lambda_{1,p}(S_c) \leq \lambda_{1,p}(A) = \lambda,$$



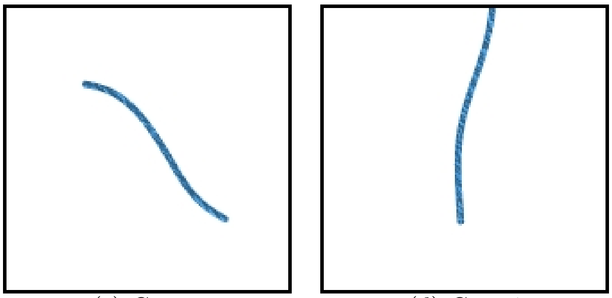
(a) Case 1 (b) Case 2

yielding that

$$\lambda_{1,p}(S_1) \leq Area(\Omega)^p,$$

a contradiction (simply rescale Ω if necessary).

Consider the two cases showed in Figure 3. We show that these cases are also not possible.



(c) Case 3 (d) Case 4

By contradiction, suppose that case 3 is possible. Let A denote the nodal domain containing such nodal line. Restrict the eigenfunction to A. Since the nodal line does not touch the boundary of A, the Theorem 5.4.1 yields a contradiction. Indeed, consider $K = K(3\rho) \subset \Omega$ not touching the boudary including a part of the nodal line. Theorem 5.4.1 yields that $u_\lambda \equiv 0$, a contradiction.

By contradiction, suppose that case 4 is possible. In order for Theorem 5.4.1 to fail, we know that every neighborhood of the boundary of such line must contain points such that u_λ changes sign. To do so, the nodal line must be closed, yielding a contradiction.

Therefore, the first case leads to the lower bound $C_1\lambda^{-1/p}$ for the length of the nodal lines in each square. Since $Area(\Omega) = Area(S_c)\lambda^{2/p}$, there are roughly $\lambda^{2/p}$ squares covering Ω, implying that

$$\mathcal{H}^1(\mathcal{Z}_\lambda) \geq C_2\lambda^{2/p}\lambda^{-1/p} = C_2\lambda^{1/p}.$$

5.5. **Proof of Theorem 3.2.5.** Using [Maz, Theorem 14.1.2], we get the following lemma:

Lemma 5.5.1. *Let F be a compact subset of $\overline{B}_r$.*

1. If $1<p<n$, for all $u\in C^\infty(\overline{B}_r)$ such that $u\equiv 0$ on F, there exists a positive constant $C_1(n,p)$ depending only on n and p such that

$$cap_p(F)\leq \frac{C_1(n,p)}{r^{-n}}\frac{\int_{\overline{B}_r}|\nabla u|^p}{\int_{\overline{B}_r}|u|^p}. \tag{5.5.2}$$

2. If $1<p<n$, for all $u\in C^\infty(\overline{B}_r)$ such that $u\equiv 0$ on F, where F is a negligible subset of $\overline{B}_r$, and

$$||u||_{L^p(\overline{B}_{r/2})}\leq C||\nabla u||_{L^p(\overline{B}_r)}, \tag{5.5.3}$$

then there exists a positive constant $C_2(n,p)$ depending only on n and p such that

$$cap_p(F)\geq \frac{C_2(n,p)}{r^{-n}}\frac{\int_{\overline{B}_r}|\nabla u|^p}{\int_{\overline{B}_r}|u|^p}. \tag{5.5.4}$$

Note that the restriction $1<p<n$ mainly comes from the Sobolev Embedding Theorem, which is needed to prove [Maz, Theorem 14.1.2]. Let us also recall the following two properties of the p-capacity (which are proved in [Maz]):

- The p-capacity is monotone: $F_1\subset F_2 \implies \mathrm{cap}_p(F_1)\leq \mathrm{cap}_p(F_2)$.
- The p-capacity of a closed ball of radius r can be computed explicitly:

$$\mathrm{cap}_p(\overline{B}_r)=r^{n-p}\mathrm{cap}_p(\overline{B}_1)=r^{n-p}\omega_n\left(\frac{|n-p|}{p-1}\right)^{p-1}, \tag{5.5.5}$$

  where ω_n is the area of the unit sphere $S^{n-1}\subset\mathbb{R}^n$ and $1<p<n$.

We are ready to begin the proof of the lower bound. The ideas used in the following proof comes from [MS].

Lower bound of Theorem 3.2.5. Fix $\gamma\in(0,1)$ and choose any $r>r_{\Omega,\gamma}$. Then, any ball $\overline{B}_r$ is of non-negligible intersection, i.e.

$$\mathrm{cap}_p(\overline{B}_r\setminus\Omega)\geq\gamma\mathrm{cap}_p(\overline{B}_r).$$

Since any $u\in C_0^\infty(\Omega)$ vanishes on $\overline{B}_r\setminus\Omega$, one can use Lemma 5.5.1, part 1. Using (5.5.6) and the explicit value of the p-capacity of a closed ball, one gets the following:

$$\begin{aligned}\int_{\overline{B}_r}|u|^p dx &\leq \frac{C_1(n,p)}{r^{-n}\mathrm{cap}_p(\overline{B}_r\setminus\Omega)}\int_{\overline{B}_r}|\nabla u|^p dx\\ &\leq \frac{C_1(n,p)}{r^{-n}\gamma\mathrm{cap}_p(\overline{B}_r)}\int_{\overline{B}_r}|\nabla u|^p dx\\ &\leq \frac{C_1(n,p)}{r^{-p}\gamma\mathrm{cap}_p(\overline{B}_1)}\int_{\overline{B}_r}|\nabla u|^p dx.\end{aligned} \tag{5.5.6}$$

Choose a covering of $\mathbb{R}^n$ by balls $\overline{B}_r=\overline{B}_r^{(k)}$, $k=1,2,...$, so that the multiplicity of this covering is at most $N=N(n)$, which is bounded since for example, for $n\geq 2$, the following estimate is valid (see [R, Theorem 3.2]):

$$N(n)\leq n\log(n)+n\log(\log(n))+5n.$$

Sum up (5.5.6) to get the following:

$$\begin{aligned}\int_{\mathbb{R}^n} |u|^p dx &\leq \sum_k \int_{\overline{B}_r^{(k)}} |u|^p dx \\ &\leq \frac{C_1(n,p)}{r^{-p}\gamma \mathrm{cap}_p(\overline{B}_1)} \sum_k \int_{\overline{B}_r^{(k)}} |\nabla u|^p dx \\ &\leq \frac{C_1(n,p)N(n)}{r^{-p}\gamma \mathrm{cap}_p(\overline{B}_1)} \int_{\mathbb{R}^n} |\nabla u|^p dx.\end{aligned}$$

Since for all $u \in C_0^\infty(\Omega)$, we have that

$$\frac{\gamma \mathrm{cap}_p(\overline{B}_1) r^{-p}}{C_n N(n)} \leq \frac{\int_{\mathbb{R}^n} |\nabla u|^p dx}{\int_{\mathbb{R}^n} |u|^p dx};$$

therefore, we get that

$$\lambda_{1,p}(\Omega) \geq K_1(\gamma, n, p) r^{-p} = \frac{\gamma \mathrm{cap}_p(\overline{B_1})}{C_1(n,p)N(n)} r^{-p}. \tag{5.5.7}$$

Taking the limit of (5.5.7) as $r \searrow r_{\Omega,\gamma}$ yields the desired result. $\square$

The proof of the upper bound is very similar to the last one, but uses the second part of Lemma 5.5.1. However, this proof is different from the one given in [MS], but has the disadvantage of not yielding an explicit constant, but no such constant are known in the case of the p-Laplacian (recall that Lieb's constant for the lower bound and that the upper bound given in (2.1.1) are not totally explicit since they both depend on $\lambda_{1,p}(B)$).

Upper bound of Theorem 3.2.5. Fix $\gamma \in (0,1)$. Consider $r_{\Omega,\gamma}$. By definition, we know that

$$\mathrm{cap}_p(\overline{B}_{r_{\Omega,\gamma}} \setminus \Omega) \leq \gamma \mathrm{cap}_p(\overline{B}_{r_{\Omega,\gamma}}).$$

We know want to use Lemma 5.5.1, part 2. Let $F = \overline{B}_{r_{\Omega,\gamma}} \setminus \Omega$. Clearly, F is a negligible subset of $\overline{B}_{r_{\Omega,\gamma}}$. It is also clear that any test function $u \in C_0^\infty(\Omega)$ will vanish identically on F. Therefore, for any such function, using Poincaré inequality, we get

$$||u||_{L^p(\overline{B}_{r_{\Omega,\gamma}/2})} \leq ||u||_{L^p(\overline{B}_{r_{\Omega,\gamma}})} \leq C||\nabla u||_{L^p(\overline{B}_{r_{\Omega,\gamma}})}.$$

Therefore, one can use Lemma 5.5.1 part 2 and get :

$$\begin{aligned}\int_{\overline{B}_{r_{\Omega,\gamma}}} |u|^p dx &\geq \frac{C_2(n,p)}{r_{\Omega,\gamma}^{-n} \mathrm{cap}_p(\overline{B}_{r_{\Omega,\gamma}} \setminus \Omega)} \int_{\overline{B}_{r_{\Omega,\gamma}}} |\nabla u|^p dx \\ &\geq \frac{C_2(n,p)}{r_{\Omega,\gamma}^{-n} \gamma \mathrm{cap}_p(\overline{B}_{r_{\Omega,\gamma}})} \int_{\overline{B}_{r_{\Omega,\gamma}}} |\nabla u|^p dx \\ &\geq \frac{C_2(n,p)}{r_{\Omega,\gamma}^{-p} \gamma \mathrm{cap}_p(\overline{B}_1)} \int_{\overline{B}_{r_{\Omega,\gamma}}} |\nabla u|^p dx.\end{aligned}$$

Choose a covering of $\mathbb{R}^n$ by balls $\overline{B}_{r_{\Omega,\gamma}} = \overline{B}^{(k)}_{r_{\Omega,\gamma}}$, $k = 1, 2, ...$, so that the multiplicity of this covering is at most $N = N(n)$, and get

$$\begin{aligned}\int_{\mathbb{R}^n} |\nabla u|^p dx &\leq \sum_k \int_{\overline{B}^{(k)}_{r_{\Omega,\gamma}}} |\nabla u|^p dx \\ &\leq \frac{r_{\Omega,\gamma}^{-p} \gamma \mathrm{cap}_p(\overline{B}_1)}{C_2(n,p)} \sum_k \int_{\overline{B}^{(k)}_{r_{\Omega,\gamma}}} |u|^p dx \\ &\leq \frac{r_{\Omega,\gamma}^{-p} \gamma \mathrm{cap}_p(\overline{B}_1) N(n)}{C_2(n,p)} \int_{\mathbb{R}^n} |u|^p dx.\end{aligned}$$

For such $u \in C_0^\infty(\Omega)$, we have that

$$\lambda_{1,p}(\Omega) \leq \frac{\int_{\mathbb{R}^n} |\nabla u|^p dx}{\int_{\mathbb{R}^n} |u|^p dx} \leq \frac{r_{\Omega,\gamma}^{-p} \gamma \mathrm{cap}_p(\overline{B}_1) N(n)}{C_2(n,p)};$$

thus, yielding the desired result with $K_2(\gamma, n, p) = \frac{\gamma \mathrm{cap}_p(\overline{B}_1) N(n)}{C_2(n,p)}$. □

5.6. **Proofs of Theorem 4.2.1 and of Propositions 4.2.3, 4.3.1.** The proofs are very straightforward and consist of combining Lemma 2.2.1 with use [O3, p.26, eq. (121)-(122)], [O2, p. 1208, Eq. (4.31)], or [O4, Eq. (14)] respectively.

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